2/18/26

Exam 1 next Wed 2/25.

Closed book - No notes.

Covers all of ch1, ch2. None of ch 3. (Although we'll start Ch 3 Wednesday 2/18.)

No calculators! Cell phones should be put away.

Practice exam + solutions are on webpage.

Today start Ch 3: Image + kernel of a linear transf.

The def. of image holds for functions in general. The def of kernel is more special, but also could be defined in a more general context (but we won't).

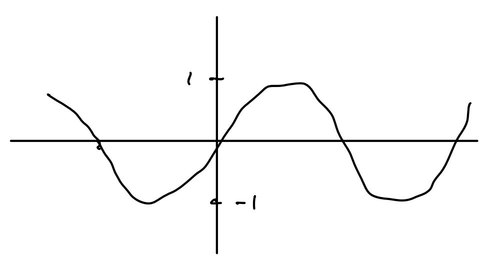
Start with image.

Def Let $f: A \to B$ be a function (so A is its domain). The image of f, denoted image(f), is the subset of B defined by

$$\text{image}(f) = \{ f(x) \mid x \in A \}$$
$$= \{ \text{elements of } B \text{ that are "hit" by } f \}$$
$$= \{ b \in B \mid b = f(x) \text{ for some } x \in A \}.$$

Ex $f: \mathbb{R} \to \mathbb{R}$

$f(x) = \sin x$ (this is not linear)



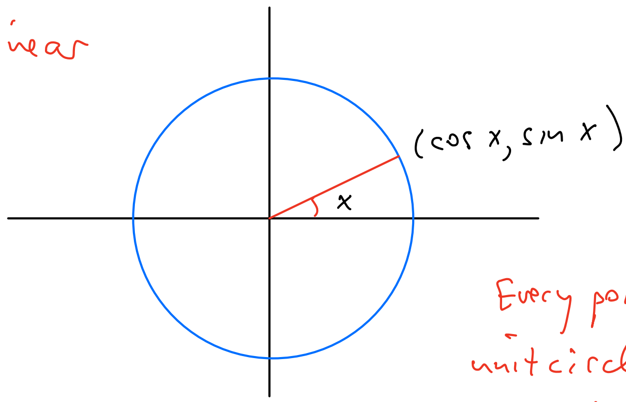
image$(f) = [-1, 1]$

closed interval from -1 to 1

Ex $f: \mathbb{R} \to \mathbb{R}^2$

$$f(x) = \begin{bmatrix} \cos x \\ \sin x \end{bmatrix}$$

Again this is not linear



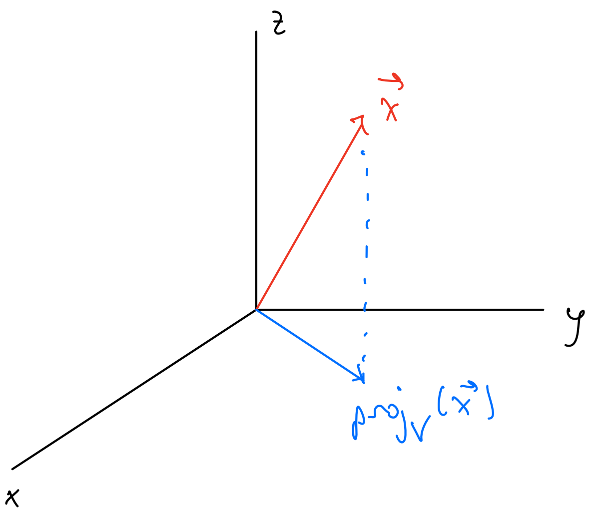
image(f) = unit circle

Note $\sin^2 x + \cos^2 x = 1$

Every point on the unit circle is of the form $(\cos x, \sin x)$

Now focus on linear transformations.

Ex in $\mathbb{R}^3$, $T(\vec{x}) = \text{proj}_V(\vec{x})$ where V is the (x,y)-plane



image(T) = V

= (x,y)-plane

Ex $T: \mathbb{R}^3 \to \mathbb{R}^3$ given by mult. by $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

So

$$\begin{bmatrix} T(\vec{e}_1) \mid T(\vec{e}_2) \mid T(\vec{e}_3) \end{bmatrix} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

$$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad T(\vec{e}_2) = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \qquad T(\vec{e}_3) = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

$\Rightarrow$ (for example)

$$T\left(\begin{bmatrix}10\\11\\12\end{bmatrix}\right) = T\left(10\vec{e}_1 + 11\vec{e}_2 + 12\vec{e}_3\right)$$

$$= 10\,T(\vec{e}_1) + 11\,T(\vec{e}_2) + 12\,T(\vec{e}_3)$$

$$= 10\begin{bmatrix}1\\2\\3\end{bmatrix} + 11\begin{bmatrix}4\\5\\6\end{bmatrix} + 12\begin{bmatrix}7\\8\\9\end{bmatrix} = \begin{bmatrix}138\\171\\204\end{bmatrix}$$

So image(T) = all possible linear comb. of $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}$.

= span (columns of A)

More precisely, given vectors $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^n$, their span is

$$\text{span}(\vec{v}_1, \dots, \vec{v}_m) = \{a_1\vec{v}_1 + \dots + a_m\vec{v}_m \mid a_1, \dots, a_m \in \mathbb{R}\}.$$

So in our example,

image(T) = span of the column vectors of A

This is also called the <u>column space</u> of $A = \begin{bmatrix}1 & 4 & 7\\2 & 5 & 8\\3 & 6 & 9\end{bmatrix}$.

Note- You can check that in <u>this</u> example A is invertible.

$\Rightarrow$ $A\vec{x} = \vec{b}$ is consistent $\forall\, b \in \mathbb{R}^3$

$\Rightarrow$ every $\vec{b} \in \mathbb{R}^3$ is a lin. comb. of the columns of A

$\Rightarrow$ image(T) is all of $\mathbb{R}^3$.